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A Generalization of the K-V Equilibrium For An Intense, Relativistic Beam Propagating in a Solenoid

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A generalization of the Kapchinskij-Vladimirskij (K-V) equilibrium for an electron beam in a solenoid is derived by keeping higher order terms in an expansion of a Vlasov equilibrium. The radial profile of the charge density increases with r, whereas the radial profile of the K-V is flat. Space charge depression effects, the shear in longitudinal momentum and the self-induced longitudinal diamagnetic field are included. Numerical simulations of the beam transport in the Spiral Line Induction Accelerator (SLIA) illustrate the results.

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CONTENTS

A GENERALIZATION OF THE K-V EQUILIBRIUM FOR AN INTENSE RELATIVISTIC BEAM PROPAGATING IN A SOLENOID

1. INTRODUCTION

The Vlasov equilibrium for a charged particle beam in a uniform axial magnetic field with distribution function

$$f(r,p) = n_0/2\pi\gamma_0 m \; \delta(H-\omega L-\gamma_0 mc^2) \; \delta(P-\beta_0\gamma_0 mc)$$

is examined, where n_0, γ_0, ω and β_0 are constants. An expansion for which the dimensionless quantities $4\pi b^2e^2n_0/\gamma_0 mc^2$, $\omega b/c$, and $1-\beta_0{}^2-1/\gamma_0{}^2$ are small is taken where b is an order-one scale length. In the limit that $\gamma_0 \to \infty$ and $\omega \to 0$, the K-V distribution is recovered [1,2, pp. 588-594]. The four constants in the distribution function, n_0, γ_0, ω and β_0, are related to the more natural specification of the beam by its total current, total energy, canonical angular momentum, and perpendicular emittance, ε_n, respectively. As an example of a case where the corrections to the K-V distribution are important, we discuss the beam proposed for the Spiral Line Induction Accelerator (SLIA) Proof-of-Concept Experiment (POCE) [3].

2. CHOICE OF VLASOV EQUILIBRIUM

Consider an equilibrium which is independent of z, the propagation direction, and θ. The only fields are $E_r(r) = -d\Phi/dr$, $B_z(r) = 1/r\ drA_\theta/dr$, and $B_\theta(r) = -dA_z/dr$, where Φ is the scalar potential and A_θ and A_z are components of the vector potential. The constants of motion are

$$H = \gamma mc^2 + Ze\Phi, \qquad \text{the energy,}$$

$$L = r(p_\theta + ZeA_\theta/c), \qquad \text{the canonical angular momentum, and}$$

$$P = p_z + ZeA_z/c, \qquad \text{the canonical axial momentum,}$$

[2, pp. 84-88]. We will retain the charge number Z which is -1 for electrons. The situation we are most interested in is the description of a beam traveling in an external, uniform solenoidal field, $B_z = B_0$ for which $A_\theta^{ext}(r) = rB_0/2$.

Any distribution of the form $f(r,p) = f_0(H,L,P)$, where f_0 is a given function, is an equilibrium solution of the Vlasov equation [2]. As will be shown, the form from which the K-V can be derived is given by

$$f(r,p) = n_0/2\pi\gamma_0 m\ \delta(H-\omega L-\gamma_0 mc^2)\ \delta(P-\beta_0\gamma_0 mc) \qquad (1)$$

where n_0, ω, γ_0 and β_0 are constants.

Denote the dimensionless potentials $\phi = e\Phi/mc^2$, $a_z = eA_z/mc^2$ and $a_\theta = eA_\theta/mc^2$.

Consider an integrable function $g(r,p_r,p_\theta,p_z)$. Define $\langle g(r,p)\rangle \equiv \int g(r,p)f(r,p)d^3p$, then it can be shown that

$$\langle g\rangle = \frac{n_0}{2\pi}\int_{-\pi/2}^{\pi/2} d\tau\Big(g(r,mc\lambda'\cos(\tau),mc\lambda\sin(\tau) + mc\nu,mc(\beta_0\gamma_0-Za_z)) + g(r,-mc\lambda'\cos(\tau),mc\lambda\sin(\tau) + mc\nu,\ mc(\beta_0\gamma_0-Za_z))\Big)(\mu + \kappa\sin(\tau)) \qquad (2)$$

where

$$\lambda = \frac{\gamma_0}{1-\upsilon^2 r^2} \left(\left[1 + \frac{\upsilon r Z a_\theta}{\gamma_0} - \frac{Z\phi}{\gamma_0} \right]^2 - \left[\frac{1}{\gamma_0^2} + \left(\beta_0 - \frac{Z a_z}{\gamma_0} \right)^2 \right] \{1-\upsilon^2 r^2\} \right)^{1/2},$$

$$\lambda' = \lambda(1 - \upsilon^2 r^2)^{1/2},$$

$$\nu = \frac{\gamma_0 \upsilon r}{1-\upsilon^2 r^2} \left(1 + \frac{\upsilon r Z a_\theta}{\gamma_0} - \frac{Z\phi}{\gamma_0} \right), \tag{3}$$

$$\mu = \frac{1}{\left(1-\upsilon^2 r^2\right)^{3/2}} \left(1 + \frac{\upsilon r Z a_\theta}{\gamma_0} - \frac{Z\phi}{\gamma_0} \right),$$

$$\kappa = \frac{\upsilon r}{\left(1-\upsilon^2 r^2\right)^{3/2}} \left(\left[1 + \frac{\upsilon r Z a_\theta}{\gamma_0} - \frac{Z\phi}{\gamma_0} \right]^2 - \left[\frac{1}{\gamma_0^2} + \left(\beta_0 - \frac{Z a_z}{\gamma_0} \right)^2 \right] \{1-\upsilon^2 r^2\} \right)^{1/2},$$

and

$$\upsilon = \omega/c .$$

The assumption that $|\upsilon r| = |\omega r/c| < 1$ has been made.

The beam radial extent is determined by the region for which the argument of the square root in the expression for λ is non-negative.

We are interested in parameter regimes which allow rapidly convergent power series in the perpendicular momenta. To the lowest orders we have $\lambda^2/\gamma^2 \sim 1 - \beta_0{}^2 - 1/\gamma_0{}^2$ + order-one constant times (ϕ/γ, a_θ/γ, a_z/γ or $\upsilon^2 r^2$) and $\nu^2/\gamma^2 \sim \upsilon^2 r^2$. For relativistic, low emittance beams the first term is small because $\beta_0 \approx 1$ and $\gamma_0 \gg 1$. However, an expansion in powers of the perpendicular momentum is also valid when $\gamma_0 \to 1$ and $\beta_0 \to 0$ at least for low charge density beams where ϕ/γ_0, a_z/γ_0 and a_θ/γ_0 are small.

Note that $\langle g \rangle = 0$ for all odd functions of p_r.

3. EXAMPLES OF MOMENTS OF THE DISTRIBUTION

1) Particle density: $n_e = \langle 1 \rangle$ gives

$$n_e = \frac{n_0}{\left(1-\upsilon^2 r^2\right)^{3/2}} \left(1 + \frac{\upsilon r Z a_\theta}{\gamma_0} - \frac{Z\phi}{\gamma_0} \right) . \tag{4}$$

This result is exact.

For $\omega=0$ and $f(r,p) \propto \delta(H-\gamma_0 mc^2)$ the result is $n_e = n_0\ (1 - Z\phi/\gamma_0)$ [4, 2, pp. 122-130] and the scalar potential is a modified Bessel function out to the beam radius. The "nonrelativistic" or "rigid-rotor" case is recovered if the υ terms are dropped and low current is assumed so that $Z\phi/\gamma_0$ is ignored. This results in the flat-top radial density of the K-V distribution [2, pp. 588-594]. The beams under study will not be flat-top, but they will have a sharp radial cutoff.

2) Moments of the momenta: All integral powers of the momenta can be integrated exactly. The ones of immediate interest are

$$\begin{aligned}
\langle p_\theta \rangle/mc &= n_0\ (\lambda\kappa/2 + \mu\nu) \\
\langle p_\theta^2 \rangle/m^2c^2 &= n_0\ (\lambda^2\mu/2 + \lambda\nu\kappa + \nu^2\mu) \\
\langle p_\theta^3 \rangle/m^3c^3 &= n_0\ (3\lambda^3\kappa/8 + 3\lambda^2\nu\mu/2 + 3\lambda\nu^2\kappa/2 + \nu^3\mu) \\
\langle p_r^2 \rangle/m^2c^2 &= n_0\ \lambda'^2\mu/2 \ = n_0(1-\upsilon^2r^2)\lambda^2\mu/2 \\
\langle p_\theta p_r^2 \rangle/m^3c^3 &= n_0\ (\lambda\lambda'^2\kappa/8 + \lambda'^2\nu\mu/2) = n_0(1-\upsilon^2r^2)(\lambda^3\kappa/8 + \lambda^2\mu\nu/2).
\end{aligned} \tag{5}$$

Moments involving γ result in elliptic integrals.

Substituting from Eq. (3), we get

$$\langle p_\theta\rangle/\gamma_0 mc = n_0 \upsilon r\,(1-\upsilon^2r^2)^{1/2}\left(\frac{\lambda^2}{2\gamma_0^2}\frac{1}{(1-\upsilon^2r^2)} + \mu^2\right) \tag{6}$$

$$= \frac{3n_0\upsilon r}{2(1-\upsilon^2r^2)^{5/2}}\left[1 + \frac{\upsilon rZa_\theta}{\gamma_0} - \frac{Z\phi}{\gamma_0}\right]^2 - \frac{n_0\upsilon r}{2(1-\upsilon^2r^2)^{3/2}}\left[\frac{1}{\gamma_0^2} + \left(\beta_0 - \frac{Za_z}{\gamma_0}\right)^2\right].$$

Consider the canonical angular momentum density,

$$\frac{\langle L\rangle}{\gamma_0 mc} = \frac{r\langle p_\theta\rangle}{\gamma_0 mc} + \frac{rZa_\theta n_e}{\gamma_0} = r^2n_0\upsilon + rn_0\frac{Za_\theta}{\gamma_0} + \text{higher order terms.}$$

In a solenoid $ra_\theta = r^2eB_0/2mc^2$ if the self-fields are neglected. In the case of a beam created in a diode free of magnetic field, the canonical angular momentum is zero. This is assured, to lowest order, by $\omega/c = \upsilon = -ZeB_0/2mc^2\gamma_0 = B_0(\mathrm{kG})/\ 3.4\gamma_0\ \mathrm{cm}^{-1}$.

Denote $p_\perp^2 = p_r^2 + p_\theta^2$. Then

$$\langle p_\perp^2\rangle/\gamma_0^2m^2c^2 = \frac{n_0}{(1-\upsilon^2r^2)^{5/2}}\left[1 + \frac{\upsilon rZa_\theta}{\gamma_0} - \frac{Z\phi}{\gamma_0}\right]\left[\frac{2 + 3\upsilon^2r^2}{2(1-\upsilon^2r^2)}\left[1 + \frac{\upsilon rZa_\theta}{\gamma_0} - \frac{Z\phi}{\gamma_0}\right]^2 - \frac{2+\upsilon^2r^2}{2}\left[\frac{1}{\gamma_0^2} + \left(\beta_0 - \frac{Za_z}{\gamma_0}\right)^2\right]\right], \text{ and}$$

$$\langle p_\theta p_\perp^2\rangle/\gamma_0^3m^3c^3 = n_0\upsilon r\left[\frac{20 + 15\upsilon^2r^2}{8(1-\upsilon^2r^2)^{9/2}}\left[1 + \frac{\upsilon rZa_\theta}{\gamma_0} - \frac{Z\phi}{\gamma_0}\right]^4\right. \tag{7}$$

$$- \frac{12 + 3\upsilon^2r^2}{4(1-\upsilon^2r^2)^{7/2}}\left[1 + \frac{\upsilon rZa_\theta}{\gamma_0} - \frac{Z\phi}{\gamma_0}\right]^2\left[\frac{1}{\gamma_0^2} + \left(\beta_0 - \frac{Za_z}{\gamma_0}\right)^2\right]$$

$$\left. + \frac{4 - \upsilon^2r^2}{8(1-\upsilon^2r^2)^{5/2}}\left[\frac{1}{\gamma_0^2} + \left(\beta_0 - \frac{Za_z}{\gamma_0}\right)^2\right]^2\right].$$

4. EXPANSION IN THE "BEAM" APPROXIMATION

In order to compute an equilibrium configuration for this distribution, we need to compute several moments. Hence, we now examine the expansion of γ in an approximation relevant to beams of interest. Define $\gamma_b = (\beta_0{}^2\gamma_0{}^2 + 1)^{1/2}$ and $\beta_b = (1-1/\gamma_b{}^2)^{1/2}$. It follows that $\beta_b\gamma_b = \beta_0\gamma_0$.

In order to call this a "beam" we make the following assumptions:

$$1 - \beta_0{}^2 - 1/\gamma_0{}^2 = 1 - \gamma_b{}^2/\gamma_0{}^2 \ll 1 \; ,$$

$$(p_r{}^2 + p_\theta{}^2) \ll (\gamma_b mc)^2 \; , \quad \text{and}$$

$$\left| \, Ze(A_z, A_\theta, \Phi) \,/\, \gamma_b mc^2 \, \right| \ll 1 \; .$$

The last assumption can result from a "small" current or a "large" γ. The relativistic factor, γ, is

$$\gamma = \left(1 + \frac{p_r^2 + p_\theta^2 + (\, P - ZeA_z/c)^2}{m^2c^2} \right)^{1/2},$$

which, to first order in $(p_r^2 + p_\theta^2) \,/\, \gamma_b^2 m^2 c^2$ and a_z/γ_b , can be written

$$\gamma \approx \gamma_b \left(1 + \frac{p_r^2 + p_\theta^2}{2\gamma_b^2 m^2 c^2} - \frac{Z\beta_b a_z}{\gamma_b} \right) , \tag{8}$$

where we have used $P = \beta_0\gamma_0 mc$. To the same order,

$$\frac{1}{\gamma} \approx \frac{1}{\gamma_b} \left(1 - \frac{p_r^2 + p_\theta^2}{2\gamma_b^2 m^2 c^2} + \frac{Z\beta_b a_z}{\gamma_b} \right) ,$$

$$\frac{v_z}{c} = \frac{p_z}{\gamma mc} \approx \beta_b - \frac{\beta_b}{2} \frac{p_r^2 + p_\theta^2}{\gamma_b^2 m^2 c^2} - \frac{Za_z}{\gamma_b^3} ,$$

$$\frac{v_\theta}{c} = \frac{p_\theta}{\gamma mc} \approx \frac{p_\theta}{\gamma_b mc} \left(1 - \frac{p_r^2 + p_\theta^2}{2\gamma_b^2 m^2 c^2} + \frac{Z\beta_b a_z}{\gamma_b} \right) , \tag{9}$$

$$\frac{v_r}{c} = \frac{p_r}{\gamma mc} \approx \frac{p_r}{\gamma_b mc}\left(1 - \frac{p_r^2 + p_\theta^2}{2\gamma_b^2 m^2 c^2} + \frac{Z\beta_b a_z}{\gamma_b}\right), \text{ and}$$

$$\frac{H - \omega L}{mc^2} \approx \gamma_b + Z\phi - Z\beta_b a_z - \frac{Zr\omega a_\theta}{c} - \frac{r^2\omega^2\gamma_b}{2c^2} + \frac{1}{2}\,\frac{p_r^2 + (p_\theta - r\omega\gamma_b m)^2}{\gamma_b\, m^2 c^2}\,.$$

Note that by keeping the self-field corrections in γ, the mean azimuthal velocity $V_\theta = \langle v_\theta\rangle/\langle 1\rangle$ is not exactly $r\omega$ (referred to as "rigid-rotor" equilibria).

5. CALCULATION OF SOURCE TERMS FOR MAXWELL'S EQUATIONS

We calculate the charge and current densities to the order indicated above: we retain terms , ϕ/γ, a_z/γ, $\upsilon b a_\theta/\gamma$, υb, $\upsilon^2 b^2$, $\upsilon b\phi/\gamma$, $\upsilon b a_z/\gamma$, $\upsilon^2 b^2 a_\theta/\gamma$, $\upsilon^3 b^3$, $(1 - \gamma_b^2/\gamma_0^2)$, and $\upsilon b(1 - \gamma_b^2/\gamma_0^2)$ where b is a characteristic length of the same order as the radial dimension for the beam. The value of b is unimportant, as it does not appear in the final equations.

1. Charge density, $\rho = Ze\langle 1\rangle$:

$$\rho = Zen_0\left(1 + \frac{3\upsilon^2 r^2}{2} + \frac{\upsilon r Z a_\theta}{\gamma_0} - \frac{Z\phi}{\gamma_0}\right). \tag{10}$$

2. Axial current density, $J_z = Ze\langle v_z\rangle$:

$$J_z = Zecn_0\left(\beta_b\left(\frac{3}{2} - \frac{\gamma_0^2}{2\gamma_b^2}\right) + \frac{\beta_b\upsilon^2 r^2}{2}\left(6 - \frac{5\gamma_0^2}{\gamma_b^2}\right) - \frac{Za_z}{\gamma_b} + \frac{3\beta_b}{2}\left(\frac{\upsilon r Z a_\theta}{\gamma_0} - \frac{Z\phi}{\gamma_0}\right)\left(1 - \frac{\gamma_0^2}{\gamma_b^2}\right)\right)$$

$$= Zecn_0\left(\beta_0 + \frac{\beta_0\upsilon^2 r^2}{2} - \frac{Za_z}{\gamma_0}\right). \tag{11}$$

3. Azimuthal current density, $J_\theta = Ze\langle v_\theta \rangle$:

$$
\begin{aligned}
J_\theta = Zecn_0\upsilon r \Bigg[& \left(\frac{3\gamma_0}{\gamma_b} - \frac{3\gamma_b}{4\gamma_0} - \frac{5\gamma_0^3}{4\gamma_b^3} \right) + \frac{3\beta_b Za_z}{2\gamma_0}\left(1 - \frac{\gamma_0^2}{\gamma_b^2}\right) \\
& + \frac{\upsilon^2 r^2}{16}\left(24 - 45\left(1 - \frac{\gamma_0^2}{\gamma_b^2}\right) + 105\frac{\gamma_0}{\gamma_b}\left(1 - \frac{\gamma_0^2}{\gamma_b^2}\right) \right) \\
& + \frac{\gamma_0}{\gamma_b}\left(\frac{\upsilon r Za_\theta}{\gamma_0} - \frac{Z\phi}{\gamma_0}\right)\left(6 - \frac{5\gamma_0^2}{\gamma_b^2}\right) \Bigg] \qquad (12) \\
\approx Zecn_0\upsilon r & \left(1 + \frac{3\upsilon^2 r^2}{2} + \left(\frac{\upsilon r Za_\theta}{\gamma_0} - \frac{Z\phi}{\gamma_0}\right) \right) \approx \rho\upsilon rc \, .
\end{aligned}
$$

If only the first term is kept, we obtain $J_\theta = Zecn_0\upsilon r$, ra_θ becomes a fourth-order polynomial in r, and ϕ and a_z are the sums of fourth-order polynomials and modified Bessel functions (inside the beam). For $\upsilon = 0$, the expression for J_z reduces to the correct expression to the proper order in the expansion [4,2, pp. 122-130]. For the K-V distribution, only the first terms in the expressions for ρ, J_z and J_θ are retained [2, pp. 588-594].

Let us introduce some dimensionless quantities. The dimensionless radial coordinate is r/b. The dimensionless particle density is $\tilde{n}_0 \equiv 4n_0\pi b^2 r_c = n_0 b(\mathrm{cm})^2/2.83\mathrm{e}11$ where $r_c = e^2/mc^2$ is the classical electron radius (a uniform 1 kA beam of 1 cm radius has a particle density n = 6.63e10 cm^{-3}). The dimensionless angular parameter is $\tilde{\omega} \equiv \omega b/c$. The dimensionless external solenoid field is $\tilde{B}_0 \equiv B_0 eb/mc^2 = B_0 b/1.7\mathrm{kG}$. The dimensionless current density is $\tilde{J} \equiv J4\pi eb^2/mc^3$ and the dimensionless beam current is $\tilde{I} \equiv Ie/mc^3 = I(\mathrm{kA})/17$. Now the equations will use dimensionless quantities and drop the "~" and we will use ω instead of υ.

6. POWER SERIES SOLUTION FOR THE POTENTIALS

We now use the power series expansions of the potentials: $\phi = \Sigma s_n r^{2n}$, $a_z = \Sigma t_n r^{2n}$ and $ra_\theta = \Sigma u_n r^{2n}$. Setting the potentials equal to 0 at the origin allows the sums to run from n=1 to n=∞.

We also define coefficients a_n, b_n and c_n for n=1 to 5 from Maxwell's equations for the potentials inside the beam:

$$\frac{1}{r}\frac{d}{dr} r \frac{d\phi}{dr} = -\rho = a_1 + a_2\phi + a_3a_z + a_4a_\theta + a_5r^2,$$

$$\frac{1}{r}\frac{d}{dr} r \frac{da_z}{dr} = -j_z = b_1 + b_2\phi + b_3a_z + b_4a_\theta + b_5r^2, \quad \text{and} \tag{13}$$

$$\frac{d}{dr}\frac{1}{r}\frac{d}{dr} ra_\theta = -j_\theta = r(c_1 + c_2\phi + c_3a_z + c_4a_\theta + c_5r^2) ,$$

where ρ is the dimensionless charge density with en_0 replaced by dimensionless n_0, υ replaced by dimensionless $\omega = \upsilon b$ and r replaced by r/b in Eq. (10). Similar substitutions are made in the previous expressions for $J_z \to j_z$ and $J_\theta \to j_\theta$.

The following equations for the coefficients result:

$$4s_1 = a_1 \qquad = -Zn_0 ,$$

$$4t_1 = b_1 \qquad = -Zn_0\beta_0 ,$$

$$16s_2 = a_2s_1 + a_3t_1 + a_4u_1 + a_5 \qquad = Z^2n_0s_1/\gamma_0 - Z^2n_0\omega u_1/\gamma_0 - 3Zn_0\upsilon^2/2 ,$$

$$16t_2 = b_2s_1 + b_3t_1 + b_4u_1 + b_5 \qquad = Z^2n_0t_1/\gamma_0 - Zn_0\beta_0\upsilon^2/2 ,$$

$$8u_2 = c_1 \qquad = -Zn_0\omega ,$$

$$24u_3 = c_2s_1 + c_3t_1 + c_4u_1 + c_5 \quad = Z^2n_0\omega s_1/\gamma_0 - Z^2n_0\omega^2u_1/\gamma_0 - 3Zn_0\omega^3/2 \ ,$$

$$36s_3 = a_2s_2 + a_3t_2 + a_4u_2 \quad = Z^2n_0s_2/\gamma_0 - Z^2n_0\omega u_2/\gamma_0 \ ,$$

and

$$36t_3 = b_2s_2 + b_3t_2 + b_4u_2 \quad = Z^2n_0t_2/\gamma_0 \ . \tag{14}$$

Note that all the coefficients are determined explicitly except u_1, which will be found later. Coefficients of order n depend on those of lower order. Specifically, $s_n \sim n_0/\gamma\ (s_{n-1} + \omega u_{n-1})$, $t_n \sim n_0/\gamma\ t_{n-1}$, and $u_n \sim n_0\omega/\gamma\ (s_{n-2}+\omega u_{n-2})$. Therefore the power series converge rapidly and can be terminated provided $n_0/\gamma_0 \ll 1$. Note that this condition, which is essentially the requirement for ignoring nonlinear terms in the potentials, is different than the condition for rapid convergence in expansions in perpendicular momenta.

From now on we consider the case of an external solenoid of (dimensionless) field B_0 and replace u_1 in the above equations with $u_1 + B_0/2$ where u_1 from now on refers only to the self-field contribution.

We obtain

$$ra_\theta = u_1r^2 + \frac{B_0r^2}{2} - \frac{Zn_0\omega}{8}r^4 - \left(\frac{Z^3n_0^2\omega}{96\gamma_0} + \frac{Z^2n_0\omega^2u_1}{24\gamma_0} + \frac{Z^2n_0\omega^2B_0}{48\gamma_0} + \frac{Zn_0\omega^3}{16}\right)r^6 \ ,$$

$$\phi = \frac{-Zn_0}{4}r^2 + \frac{-Z^3n_0^2 - 6Zn_0\omega^2\gamma_0 - 4Z^2n_0\omega u_1 - 2Z^2n_0\omega B_0}{64\gamma_0}r^4 \ , \text{ and} \tag{15}$$

$$a_z = \frac{-Zn_0\beta_0}{4}r^2 + \left(\frac{-Z^3n_0^2\beta_0}{64\gamma_0} - \frac{Zn_0\beta_0\ \omega^2}{32}\right)r^4 \ .$$

We will need the following expressions to the proper order

$$\frac{\omega r a_\theta}{\gamma_0} = \frac{\omega u_1 r^2}{\gamma_0} + \frac{\omega B_0 r^2}{2\gamma_0} ,$$

$$\frac{\phi}{\gamma_0} = \frac{-Zn_0 r^2}{4\gamma_0} - \frac{Z^2 n_0 \omega u_1 r^4}{16\gamma_0^2} - \frac{Z^2 n_0 \omega B_0 r^4}{32\gamma_0^2} , \text{ and} \tag{16}$$

$$\frac{a_z}{\gamma_0} = \frac{-Zn_0 \beta_0 r^2}{4\gamma_0} .$$

To lowest order we expect $B_0/\gamma_0 \sim \omega$, so the terms in ϕ/γ containing r^4 can usually be ignored.

7. EXPRESSION FOR THE BEAM RADIUS

The beam edge radius R is obtained by finding the domain where the argument of the square root in the expression for λ in Eq. (3), is positive. To this order the beam is confined to spatial regions satisfying

$$1 + \frac{2Z\omega r a_\theta}{\gamma_0} - \frac{2Z\phi}{\gamma_0} - \frac{\gamma_b^2}{\gamma_0^2} + \frac{2Z\beta_0 a_z}{\gamma_0} + \omega^2 r^2 \geq 0 \text{ , and}$$

we assume $1 - \gamma_b^2/\gamma_0^2 > 0$ so that the beam is not hollow. The outer cutoff radius, R, of the beam is then the solution of

$$\left(1 - \frac{\gamma_b^2}{\gamma_0^2}\right) + R^2\left(\omega^2 + \frac{2Z}{\gamma_0}\left(\omega u_1 + \frac{\omega B_0}{2} + \frac{Zn_0}{4}\left[1 - \beta_0^2\right]\right)\right)$$

$$+ R^4 \frac{Z^3 n_0 \omega}{8\gamma_0^2}\left(u_1 + \frac{B_0}{2}\right) = 0 . \tag{17}$$

8. CONSIDERATION OF CANONICAL ANGULAR MOMENTUM

The canonical angular momentum is $L^*(r) = \langle L \rangle = \langle r(p_\theta + ZeA_\theta/c)\rangle$. Expanding this, we obtain

$$L^*/n_0\gamma_0 mc = \omega r^2(3/2 - \gamma_b{}^2/2\gamma_0{}^2 + 3\omega^2 r^2 + 3Z\omega r a_\theta/\gamma_0$$

$$- 3Z\phi/\gamma_0 + \beta_0 Z a_z/\gamma_0) + Zra_\theta/\gamma_0$$

$$L^*/n_0\gamma_0 mc = r^2(\ \omega(3/2 - \gamma_b{}^2/2\gamma_0{}^2) + Zu_1/\gamma_0 + ZB_0/2\gamma_0\ ) +$$

$$r^4(\ 3\omega^3 + Z^2\omega n_0[\ 3 - \beta_0{}^2]/4\gamma_0$$

$$+ 3Z\omega^2[\ u_1 + B_0/2]/\gamma_0\ ). \tag{18}$$

For a non-immersed cathode, as is the case for the SLIA experiment,

$$\int 2\pi r L^* dr = 0. \tag{19}$$

Eq. (19) relates the parameter ω to the external field, γ_0, β_0, and u_1. Other diode configurations lead to different values for the beam's total canonical angular momentum [5, 2, pp. 559-569].

9. DETERMINATION OF THE AXIAL SELF-INDUCED MAGNETIC FIELD

The integration of Faraday's law gives the self-induced (dimensionless) axial magnetic field as

$$B_z{}^s = \int j_\theta dr = Zn_0\omega R^2/2 + 3Zn_0\omega^3 R^4/8 + Z^2 n_0 \omega R^4/4\gamma_0\ (Zn_0/4) \tag{20}$$

$$= 2u_1\ .$$

If γ_0, β_0, and n_0 are chosen, then this expression for u_1, Eq. (17) for the beam radius R, and Eqs. (18) and (19) for the zero canonical angular momentum give three (nonlinear) equations for the unknowns u_1, R, and ω, and the equilibrium is completely specified.

In the simulations to be presented later, the beam is inside a conducting cylinder of radius R_w. In this case, the relationship for u_1 is $u_1 = -u_2 R^4/R_w^2 + (1-R^2/R_w^2)\int j_\theta dr/2$. This expression accounts for the axial magnetic field produced by the image currents.

10. SPECIFICATION OF THE EQUILIBRIUM IN LABORATORY PARAMETERS

The Vlasov equilibrium was specified in terms of four parameters, n_0, γ_0, β_0 and ω with certain restrictions on their values so that the expansion is valid. Normally, however, the four parameters commonly chosen are the beam energy, beam current, emittance and canonical angular momentum.

We have seen that conditions on the canonical angular momentum are closely related to the value of ω. Clearly, beam current is correlated with n_0. In an experiment the beam electrons are created with an energy determined by the potential on the anode foil. The beam usually propagates inside a conducting cylinder, so the field energy must be taken from the beam. Note that the tube boundary, R_w, appears here if we wish to relate the diode voltage to the appropriate value of γ_0.

We define the normalized perpendicular RMS emittance, ε_n, in the relativistic setting, through

$$\varepsilon_n^2 m^2 = \langle\langle r^2\rangle\rangle\langle\langle p_r^2 + p_\theta^2\rangle\rangle - \langle\langle rp_r\rangle\rangle^2 - \langle\langle rp_\theta\rangle\rangle^2 \tag{21}$$

where

$$\langle\langle g(r,p_r,p_\theta,p_z\rangle\rangle = \int gfd^3p2\pi rdr \;/\; \int fd^3p2\pi rdr. \tag{22}$$

The emittance is related to the amount of the beam's energy found in "undirected" transverse motion [6]. The Vlasov parameter β_0 is a measure of how much of the beam's momentum is directed axially and if the other parameters are fixed, increasing β_0 will decrease the transverse emittance.

Figures 1a-c show the beam current, emittance and cutoff radius, R, as functions of the parameters in the approximate Vlasov equilibrium. We have fixed $\gamma_0 = 9$. B_0 takes the values 4.5 kG (dashed), 5.5 kG (solid), and 6.5 kG (dotted). β_0 is set to .97, .988 or .993. The assumption of zero total canonical angular momentum is made. Note that the approximations made in this model become less accurate as the parameter n_0 increases in the figures.

11. PROOF-OF-CONCEPT EXPERIMENT (POCE) IN THE SPIRAL LINE INDUCTION ACCELERATOR (SLIA)

The SLIA is a proposed compact high-current electron beam accelerator [3] which is being developed and studied experimentally at Pulse Sciences, Inc. SLIA uses induction cavities to accelerate the beam and a solenoidal magnetic field for transport in the straight sections. Transport around the bends is accomplished using strong focusing stellarator windings (i.e., twisted quadrupoles) augmented with a vertical magnetic field. The beam line is shown schematically in Fig. 2. The transport lines are isolated from each other except in the acceleration region. In the POCE a 10 kA, 35 ns beam will be accelerated to 9.5 MeV from 3.5 MeV in steps of 1.5 MeV.

The motivation for this paper came from attempts to simulate the POCE experiment [7]. Figure 3a shows a simulation of a 10.7 kA, $\gamma = 9.3$ beam with a normalized RMS emittance of .34 cm-rad in a 5.5 kG solenoidal field. The beam particles are loaded with a K-V distribution in the rotating frame with the correct moments for a matched equilibrium. Nevertheless, the beam radius exhibits growing oscillations at the envelope frequency. In this example the (transverse) emittance actually decreases slightly. After many meters the beam radius oscillations damp. This behavior indicates that the K-V equilibrium is not a good kinetic equilibrium for these beam parameters.

The next figure shows the results of a simulation with the beam initialized with the equilibrium developed in this paper. Specifically, $n_0 = 1.4\text{x}10^{12}$ per cm^3, $\gamma_0 = 9$, $\beta_0 = 0.988$ and $B_0 = 5.5$ kG. The numerical gridding and number of simulation particles is the same for the two runs. The result indicates a much better choice for initialization. For this run the cutoff radius is R = 0.70 cm. Taking the scale length as b = 0.70 cm, the dimensionless $\omega = -0.11$, dimensionless $n_0/\gamma_0 = .27$ and $1 - \beta_0{}^2 - 1/\gamma_0{}^2 = .012$. The maximum values assumed by the potentials were $\phi/\gamma_0 = .069$, $a_z/\gamma_0 = .068$, $a_\theta/\gamma_0 = .0027$ (self-potential), $b^2B_0/2\gamma_0 = .088$.

12. CONCLUSIONS

We have derived an approximate equilibrium for an intense electron beam propagating in a pure solenoidal field. This equilibrium is a higher order approximation to a true Vlasov equilibrium than the K-V equilibrium. It includes space charge depression effects, the shear in the longitudinal momentum and the self-induced longitudinal diamagnetic field. Simulations confirm the theoretical results. Contemporary and proposed experimental beams have parameters for which these effects may be measurable, even though experimental beams tend to have rounded profiles.

13. ACKNOWLEDGEMENTS

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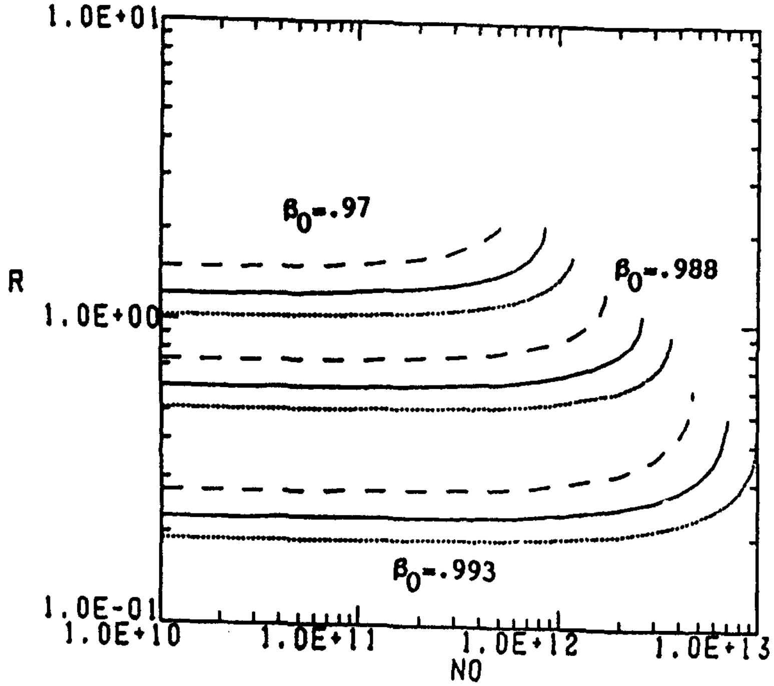

Figure 1a. Cutoff radius R (cm) as a function of n_0 (cm^{-3}) for $\gamma_0 = 9$. The other parameters are the external solenoidal field B_0 = 4.5 kG (dashed), 5.5 kG (solid), and 6.5 kG (dotted) and β_0 = .97, .988 and .993.

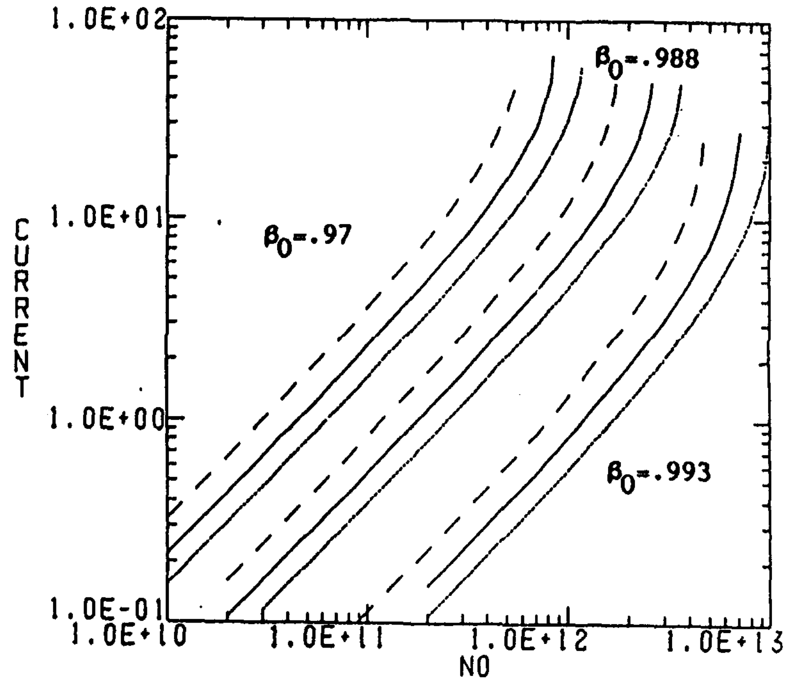

Figure 1b. Total beam current I_b (kA) as a function of n_0 (cm^{-3}) for $\gamma_0 = 9$. The other parameters are the external solenoidal field B_0 = 4.5 kG (dashed), 5.5 kG (solid), and 6.5 kG (dotted) and β_0 = .97, .988 and .993.

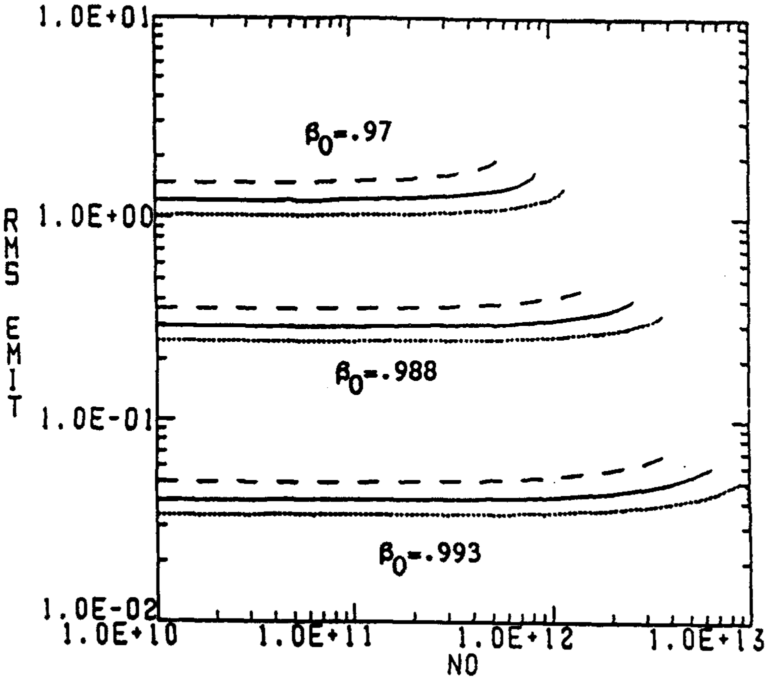

Figure 1c. Normalized RMS emittance ε_n/c (cm) as a function of n_0 (cm^{-3}) for γ_0 = 9. The other parameters are the external solenoidal field B_0 = 4.5 kG (dashed), 5.5 kG (solid), and 6.5 kG (dotted) and β_0 = .97, .988 and .993.

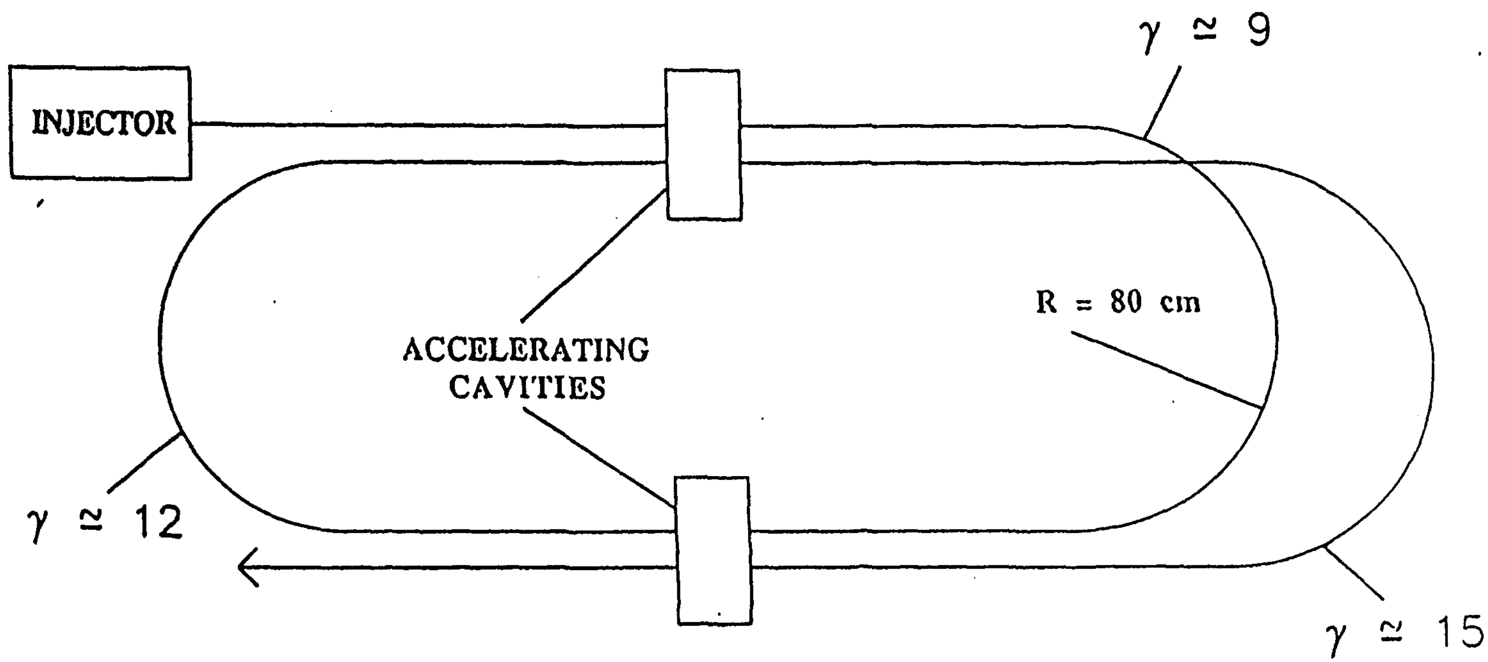

Figure 2. Beam line schematic for the Proof-of-Concept Experiment (POCE) of the Spiral Line Induction Accelerator (SLIA).

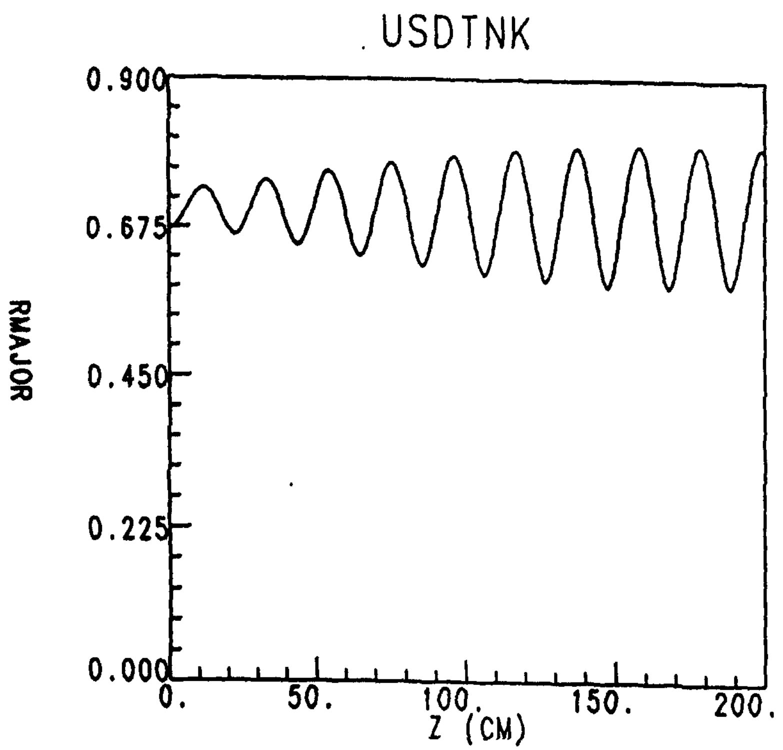

Figure 3a. Simulation plot of the cutoff radius R versus propagation distance z for a beam loaded with the K-V distribution. The current is 10.6 kA, $\gamma = 9.3$, $\varepsilon_n = .33$ cm-rad and $B_0 = 5.5$ kG ($\gamma_0 = 9$, $n_0 = 1.4\text{x}10^{12}$, $\beta_0 = .988$).

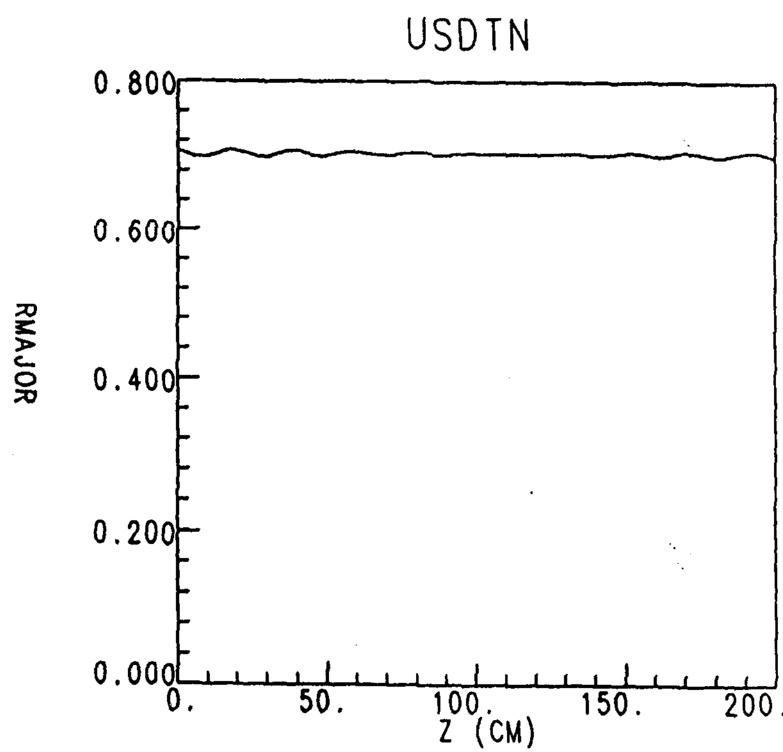

Figure 3b. Simulation plot of the cutoff radius R versus propagation distance z for a beam loaded with the distribution developed in this paper. The current is 10.6 kA, $\gamma = 9.3$, $\varepsilon_n = .33$ cm-rad and $B_0 = 5.5$ kG ($\gamma_0 = 9$, $n_0 = 1.4\text{x}10^{12}$, $\beta_0 = .988$).